STABILITY ANALYSIS OF A STEADY PLANE DETONATION WAVE

V. V. Pukhnachev

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In the present work we analyze the stability of a steady plane detonation wave within the framework of the following model of a detonation [1]: A thermally and calorically perfect gas flows at a constant supersonic speed in the direction of the axis z in the region z < 0. In the neighborhood of the plane z = 0 the flow passes through a strong shock. Downstream of the shock the gas passes through a combustion zone governed by the chemical kinetic equation [2]

$$\frac{d\beta}{dt} = -L\beta^m p^{m-1} \exp \frac{-A}{\mu p \tau}.$$

Here β is the mass concentration of reactant molecules, p is the pressure, and τ is the specific volume; the activation energy A, the mean molecular weight of the gas μ, the degree of the reaction m, and the coefficient L are positive constants and $m \geq 1$.

The chemical reaction ends when $\beta = 0$. The detonation is called a Chapman-Jouguet detonation if at $\beta = 0$ the speed of the gas equals the local speed of sound. In the model under consideration there exists a steady one-dimensional solution to the equations of hydrodynamics and kinetics

$$w = w_*(1 - cx^{-1}), \qquad p = p_*(1 + \gamma c x^{-1}),$$
$$\tau = \tau_*(1 - cx^{-1}), \qquad \beta = x^{-2} \qquad (c = (M^2 - 1)(\gamma M^2 + 1)^{-1}). \qquad (0.1)$$

Here w is the speed of the flow and M is the Mach number in the region z < 0. The subscript * denotes values at the Jouguet point. The function x = x(z) is defined by the equation

$$\int_{1/x}^{1} y^{1-2m}(1 - cy)(1 + \gamma cy)^{1-m} \exp[a(1 + \gamma cy)^{-1}(1 - cy)^{-1}]\, dy = \sigma z$$
$$\left(a = \frac{A}{\mu p_* \tau_*}, \quad \sigma = \frac{p_*^{\,m-1} L}{2w_*}\right).$$

In this paper we analyze the stability of the basic solution (0.1) to the equations of hydrodynamics and chemical kinetics with respect to small perturbations. Below we shall assume that the gas flows in a round cylindrical tube of radius r_0. We shall use a cylindrical system of coordinates with the tube axis as the z axis. Assuming that small perturbations of the flow, concentrated in the region z > 0, can be represented as a superposition of cylindrical harmonics, we shall analyze the behavior of an individual harmonic. In that case the equation of the perturbed shock surface is

$$z_0 = \varepsilon r_0 \exp(\lambda r_0^{-1} w_* t + in\varphi) J_n(\xi_{nk} r r_0^{-1}).$$

Here λ is a complex parameter, n is a natural number, ξ_{nk} is the k-th root of the equation $dJ_n(x)/dx = 0$, and $|\varepsilon| \ll 1$.

Linearizing the governing equations about the basic solution and separating the variables, we can reduce the problem of small perturbations to the following boundary-value problem for a linear system of ordinary differential equations: Find the set of functions $y_i(x)$ $(1 \leq i \leq 5)$ which satisfy over the interval $(1, \infty)$ the system of equations

$$d\mathbf{y}/dx = [\lambda s A_1(x) + \xi_{nk} s A_2(x) + A_3(x)]\mathbf{y} \qquad (0.2)$$

and the following boundary conditions.

(1) The conservation of mass, momentum, energy, and concentration across the shock, which, in terms of the functions $y_i(x)$, means

$$\mathbf{y} = \mathbf{y}_0 \equiv \lambda s \mathbf{a}_1 + \xi_{nk} s \mathbf{a}_2 + \mathbf{a}_3 \quad \text{at} \quad x = 1. \qquad (0.3)$$

(2) The boundedness of the perturbations of the velocity vector, pressure, and density and the vanishing of the perturbation of the concentration at the Jouguet point, which means

$$y_i(x) \quad (2 \leqslant i \leqslant 4), \quad \sqrt{x}\, y_1(x) \quad \text{bounded},$$
$$x^{-1} y_5(x) \to 0 \quad \text{for} \quad x \to \infty. \qquad (0.4)$$

Here the vector $\mathbf{y}(x)$ has the elements

$$y_i(x) \quad (1 \leqslant i \leqslant 5); \; s = \exp a/\sigma r_0.$$

The elements of the matrices $A_j(x)$ and of the vectors $\mathbf{a}_j$ $(1 \leq j \leq 3)$ depend on the parameters γ, c, a, and m, which shall be assumed to be constants. The expressions for $A_j(x)$ and $\mathbf{a}_j$ are quite cumbersome and we shall not write them out.

A value of λ for which the problem (0.2)–(0.4) can be solved is called an eigenvalue of the problem. The existence of even one eigenvalue with $\mathrm{Re}\,\lambda \geq 0$ would mean that the basic solution (0.1) is unstable.

An analysis of the asymptotic behavior of the solutions to system (0.2) at $x \to \infty$ leads to the following result [1]:

for $\mathrm{Re}\,\lambda \geq 0$ the system (0.2) has four linearly independent solutions which satisfy condition (0.4);

for $\mathrm{Re}\,\lambda < 0$ the system (0.2) has one solution which satisfies condition (0.4).

Thus the nature of the boundary-value problem (0.2)–(0.4) is markedly different in the cases $\mathrm{Re}\,\lambda \geq 0$ and $\mathrm{Re}\,\lambda < 0$. In the latter case the eigenvalue problem has no solution, in general, since for $\mathrm{Re}\,\lambda < 0$ the conditions (0.3), (0.4) for the fifth-order system (0.2) are equivalent to eight homogeneous boundary conditions.

The problem of finding the eigenvalues with nonnegative real part can be reduced to the solution of the equation

$$F(\lambda) \equiv \det Y(\lambda, x_0) = 0. \qquad (0.5)$$

Here the first column of the matrix Y represents the value at $x = x_0 \geq 1$ of the solution of the Cauchy problem (0.3) for the system (0.2), and the other four columns of the matrix Y are the values at $x = x_0$ of four linearly independent solutions of the system (0.2) which satisfy condition (0.4).

§1. We shall analyze the qualitative characteristics of the eigenvalue problem (0.2)–(0.4). Let M_{nk} denote the set of eigenvalues λ which correspond to fixed values of n, k, and assume $\rho = \xi_{nk} s$. Then the following assertions hold.

1°. Every set M_{nk} is bounded. The diameter d_{nk} of the set M_{nk} has the bound

$$d_{nk} \leqslant c_0 \max(\xi_{nk}, s^{-1}) \qquad (c_0 = \mathrm{const}).$$

2°. Every set M_{nk} is symmetrical with respect to the axis $\mathrm{Im}\,\lambda = 0$.

3°. Every set M_{nk} is closed, discrete, and has no points of accumulation outside the axis $\mathrm{Re}\,\lambda = 0$.

4°. The set M_{01} contains the point $\lambda = 0$ (note that $\xi_{01} = 0$).

5°. If $c \geq 1/2$, then for any $\alpha > 0$ there exists a $\rho_*(\alpha)$ such that for $\rho \geq \rho_*(\alpha)$ there are no eigenvalues in the region B_α defined by the inequalities

$$\mathrm{Re}\,\lambda \geqslant 0, \qquad |\lambda / \xi_{nk} \pm {}^1\!/_2 i \sqrt{\gamma + 1}| \geqslant \alpha.$$

6°. If $c < 1/2$, then there exists a value ρ_0 such that for $\rho \geq \rho_0$ there are no eigenvalues in the region $\mathrm{Re}\,\lambda \geq 0$.

In the following we shall prove these assertions.

Assertion 2° follows from the fact that the coefficients of the system (0.2) and the Cauchy data (0.3) are real for real λ.

To prove Assertion 3°, note that every interior point of the half-plane $\mathrm{Re}\,\lambda \geq 0$ is an interior point of one of the regions $D_{\alpha/2}$, G_α, H_α, defined respectively by

$$2|\lambda| \geqslant \xi_{nk}\alpha > 0, \qquad 2|\lambda - \xi_{nk}| \geqslant \xi_{nk}\alpha, \qquad \mathrm{Re}\,\lambda \geqslant 0,$$
$$|\lambda - \xi_{nk}| \leqslant \xi_{nk}\alpha < \xi_{nk}, \qquad |\lambda| \leqslant \xi_{nk}\alpha, \qquad \mathrm{Re}\,\lambda \geqslant 0.$$

As shown in [1], the eigenvalues $\lambda \in D_{\alpha/2}$ ($\lambda \in G_\alpha$, $\lambda \in H_\alpha$) are zeros of a function which is analytic inside the region $D_{\alpha/2}$ (G_α, H_α) and continuous on its boundary. The case $\mathrm{Re}\,\lambda < 0$ can be treated in an analogous way.

The proof of Assertion 4° is based on the fact that the system of equations of hydrodynamics and kinetics admits a translation transformation along the z axis. Applying this transformation to the basic solution (0.1), we find that the set of functions

$$u' = v' \equiv 0, \qquad w' = -cw_*x^{-2}r_0dx/dz,$$
$$p' = \gamma c p_* x^{-2} r_0 dx/dz,$$
$$\tau' = -c\tau_* x^{-2} r_0 dx/dz \qquad \beta' = 2x^{-3}r_0dx/dz \qquad (1.1)$$

represents the solution to a linearized system. This solution satisfies the conservation laws at the surface of the perturbed shock $z_0 = \varepsilon r_0$. Thus solution (1.1) represents the perturbation of the basic solution due to a small translation of the shock front along the z axis. The eigenfunction $y(x)$ of the problem (0.2)–(0.4), which corresponds to solution (1.1), has the following elements:

$$y_1 = -\frac{cx^{-2m+3/2}b(1)}{(x-c)\,b(x)}, \qquad y_2 = y_3 \equiv 0,$$
$$y_4 = \frac{c^2(\gamma+1)\,x^{-2m+1}\,b(1)}{(x-c)\,b(x)}, \qquad y_5 = \frac{2x^{-2m+2}\,b(1)}{(x-c)\,b(x)}, \qquad (1.2)$$
$$\left(b(x) = \left(\frac{x+\gamma c}{x}\right)^{1-m} \exp\left\{\frac{ac\,[-(\gamma-1)\,x+\gamma c]}{(x+\gamma c)(x-c)}\right\}\right).$$

It can be easily seen that the eigenvalue corresponding to (1.2) belongs to the set M_{01}.

We shall prove Assertion 1° together with Assertions 5° and 6°.

Assertions 5° and 6° are connected with the problem of the stability of a detonation with respect to small-scale perturbations. The point is that for fixed γ, c, a, and m there holds the relation

$$\rho = \xi_{nk}s = l\xi_{nk}\delta = ld\xi_{nk}r_0^{-1}.$$

Here d is the effective width of the chemical reaction zone [2] and l is a constant. The parameter $d\xi_{nk}r_0^{-1}$ has a simple geometrical meaning: it characterizes the ratio of the reaction zone width to the transverse scale of the shock surface perturbations. Thus the absence of eigenvalues of the problem (0.2)–(0.4) with $\mathrm{Re}\,\lambda \geq 0$ for $\rho \to \infty$ indicates the stability of the basic solution (0.1) with respect to small-scale perturbations.

The proof of Assertions 1°, 5°, and 6° is as follows. The system of equations (0.2) and conditions (0.3) for $\xi_{nk} \neq 0$ can be written in the form

$$dy/dx = \{\rho\,[\nu A_1(x) + A_2(x)] + A_3(x)\}\,y$$
$$(\nu = \lambda\xi_{nk}^{-1}), \qquad (1.3)$$

$$y = y_0 \equiv \rho\,(\nu\,a_1 + a_2) + a_3 \quad \text{at} \quad x = 1\,. \qquad (1.4)$$

Thus the problem of determining the eigenvalues $\lambda(\xi_{nk}, s)$ can be reformulated as a problem of finding the eigenvalues $\nu(\rho)$. Let us define the regions $B_{2,\alpha}$, $B_{3,\alpha}$, and $B_{1,\alpha}{}^\circ$ of the ν plane by the relations

$$1 - c - \alpha \leqslant \mathrm{Re}\,\nu \leqslant 1 + \alpha, \qquad |\mathrm{Im}\,\nu| \leqslant \alpha$$
$$(\text{region}\ B_{2,\alpha}),$$
$$0 \leqslant \mathrm{Re}\,\nu \leqslant \alpha, \qquad |\mathrm{Im}\,\nu| \leqslant {}^1/_2\sqrt{\gamma+1} + \alpha$$
$$(\text{region}\ B_{3,\alpha}),$$
$$\mathrm{Re}\,\nu \geqslant 0, \quad \nu \,\overline{\in}\, B_{2,\alpha} \cup B_{3,\alpha} \quad (0 < \alpha < {}^1/_2(1-c))$$
$$(\text{region}\ B_{1,\alpha}{}^\circ)$$

and let $B_{1,\alpha}$ denote the closure of the region $B_{1,\alpha}{}^\circ$.

We shall prove below that for any $\alpha > 0$ there exists a $\rho_1(\alpha)$ such that for $\rho \geq \rho_1(\alpha)$ there are no eigenvalues in the region $B_{1,\alpha}$. The proof of the absence of eigenvalues $\nu \in B_{2,\alpha}$ for large values of ρ is very similar to the proof of the preceding assertion. Here we shall omit the analysis of the problem (0.2)–(0.4) in the case $\rho \to \infty$, $\nu \in B_{3,\alpha}$.

Thus, let $\nu \in B_{1,\alpha}$. As shown in [1], there exists a linear transformation

$$y = M(x, \nu)\,u,$$

which is nonsingular for $x \geq 1$, $\nu \in B_{1,\alpha}$ and transforms system (0.2) to the form

$$du/dx = [\rho W(x,\nu) + P(x) + x^{-3/2}Q(x,\nu) + x^{-2}R(x,\nu)]\,u\,. \qquad (1.5)$$

Here the elements of the diagonal matrices $W(x, \nu)$, $P(x)$ are defined as

$$w_{11} = \nu b(x)\,x^{2m-3}\left\{\frac{x}{c(\gamma+1)}\,[1 + \gamma c x^{-1} + v(x,\nu)] - 1\right\},$$
$$w_{22} = \nu b(x)\,x^{2m-3}\left\{\frac{x}{c(\gamma+1)}\,[1 + \gamma c x^{-1} - v(x,\nu)] - 1\right\},$$
$$w_{jj} = -\nu b(x)\,x^{2m-3} \qquad (3 \leqslant j \leqslant 5),$$
$$(v(x,\nu) = \{(1 + \gamma c x^{-1})(1 - cx^{-1}) \times$$
$$\times [1 + c(\gamma+1)\,x^{-1}(1 - cx^{-1})\,\nu^{-2}]\}^{1/2}), \qquad (1.6)$$
$$p_{11} = {}^1/_2 x, \quad p_{55} = -(2m-1)/x,$$
$$p_{jj} = 0 \leqslant (2 \leqslant j \leqslant 4) \qquad (1.7)$$

and the elements $q_{ik}(x, \nu)$, $r_{ik}(x, \nu)$ $(1 \leq i,\ k \leq 5)$ of the matrices Q, R can be expanded in power series in x^{-1}, $x \geq 1$ for all $\nu \in B_{1,\alpha}$ and are analytic in ν inside the region $B_{1,\alpha}$ and continuous on its boundary for all $x \geq 1$. Of all the functions $q_{ik}(x, \nu)$, the only ones which are different from zero are q_{1i}, q_{i1} $(2 \le i \le 5)$; furthermore, $r_{1i}(x, \nu) = r_{i1}(x, \nu) = 0$ $(2 \le i \le 5)$.

It has been shown in [1] that if the solution $y(x, \nu)$ of the system (0.2) satisfies condition (0.4), then the solution $u(x, \nu) = M^{-1}(x, \nu)y(x, \nu)$ of system (1.5) also satisfies this condition, and vice versa. Hence it follows that the initial-value problem (0.2)–(0.4) is equivalent to the following boundary-value problem: find a vector-function $u(x, \nu)$ which over the interval $(1, \infty)$ satisfies equation (1.5) and conditions (0.4) at $x \to \infty$ and

$$u = u_0 \equiv [M(1, \nu)]^{-1} y_0(\nu) \quad \text{at } x = 1. \tag{1.8}$$

The vector u_0 can be represented in the form

$$u_0 = \rho u_1(\nu) + u_2(\nu), \tag{1.9}$$

where the elements of the vectors $\nu^{-1}u_1(\nu)$, $u_2(\nu)$ are regular in the region $B_{1,\alpha}$ and are continuous on its boundary; the function $\nu^{-1}u_{11}(\nu)$ has no zeros in the region $B_{1,\alpha}$.

In accordance with the results of [1], for any $\nu \in B_{1,\alpha}$ there exist four linearly independent solutions $u_k(x, \nu, \rho)$ $(2 \le k \le 5)$ of equation (1.5) which satisfy condition (0.4).

Consider the matrix $U(\nu, \rho)$ whose first column is the vector $u_0(\nu, \rho)$ and whose other columns are the vectors $u_k(1, \nu, \rho)$ $(2 \le k \le 5)$. It can be easily seen that the eigenvalues $\nu \in B_{1,\alpha}$ of the problem (1.5), (1.8), (0.4) are the roots of the equation

$$S(\nu, \rho) \equiv \det U(\nu, \rho) = 0. \tag{1.10}$$

We shall show that for large values of $|\rho\nu|$ this equation has no solutions in the region $B_{1,\alpha}$.

Note that for any $x \in (1, \infty)$, $\nu \in B_{1,\alpha}$ there hold the inequalities

$$\operatorname{Re}[w_{11}(x, \nu) - w_{ii}(x, \nu)] \geqslant 0,$$
$$p_{11}(x) - p_{ii}(x) \geqslant 0 \quad (2 \leqslant i \leqslant 5). \tag{1.11}$$

Using the substitution

$$u_i = \eta_{i1} \exp\left[\int_1^x (\rho w_{11} + p_{11})\, dx\right] \qquad (1 \leqslant i \leqslant 5)$$

we can reduce the system (1.5) to the system of integral equations

$$\eta_{11}(x, \nu) = 1 - \int_x^\infty \sum_{k=1}^5 f_{1k}(t, \nu)\, \eta_{k1}(t, \nu)\, dt,$$
$$\eta_{i1}(x, \nu) = \int_1^x \exp\{[\rho w_{ii}(\tau, \nu) + p_{ii}(\tau) - \rho w_{11}(\tau, \nu), - p_{11}(\tau)]\, d\tau\} \sum_{k=1}^5 f_{ik}(t, \nu)\, \eta_{k1}(t, \nu)\, dt \quad (2 \leqslant i \leqslant 5) \tag{1.12}$$
$$(f_{ik} = t^{-3/2} q_{ik} + t^{-2} r_{ik}).$$

Using (1.11) one can show that for any $\alpha > 0$ there exists an $N(\alpha)$ such that for $x \in (1, \infty)$, $\nu \in B_{1,\alpha}$, $|\rho\nu| \ge N(\alpha)$ the system (1.12) has a solution of the form

$$\eta_{i1}(x, \nu) = [\delta_{i1} + O(|\rho\nu|^{-1})] \exp\left[\int_1^x t^{-2} r_{11}(t, \nu)\, dt\right] \quad (1 \leqslant i \leqslant 5)$$

for $|\rho\nu| \to \infty$. This solution corresponds to a solution of system (1.5) which has the asymptotic representation

$$u_{i1}(x, \nu) = [\delta_{i1} + O(|\rho\nu^{-1}|)] \times \exp\left\{\int_1^x [\rho w_{11}(t, \nu) + p_{11}(t) + t^{-2} r_{11}(t, \nu)]\, dt\right\} \tag{1.13}$$

for all $x \in (1, \infty)$, $\nu \in B_{1,\alpha}$ and ρ such that $|\rho\nu| \ge N(\alpha)$ for $|\rho\nu| \to \infty$.

Let us apply to the system (1.5) the linear substitution $u = L_0 v$ where the first column of the matrix L_0 is the vector u_1 and its other columns are the vectors e_j $(2 \le j \le 5)$ (e_j is a vector whose j-th element is unity and whose other elements are all zero). The system of equations for the functions v_i $(1 \le i \le 5)$ can be reduced to the fourth-order system

$$\frac{dv_i}{dx} = [\rho w_{ii}(x, \nu) + p_{ii}(x)]\, v_i + x^{-2} \sum_{k=2}^5 r_{ik}^\circ(x, \nu)\, v_k \quad (2 \leqslant i \leqslant 5), \tag{1.14}$$
$$\left(r_{ik}^\circ(x, \nu) = r_{ik}(x, \nu) - q_{1k}(x, \nu) \frac{\sqrt{x}\, \eta_{i1}(x, \nu)}{\eta_{11}(x, \nu)} \quad (2 \leqslant i, k \leqslant 5)\right)$$

and the quadrature

$$\frac{dv_1}{dx} = u_{11}^{-1} \sum_{k=2}^5 x^{-3/2} q_{ik} v_k.$$

Note that for all $x \ge 1$ the functions $r_{ik}^\circ(x, \nu)$ are analytic in ν inside the region $B_{1,\alpha}$ and are continuous on its boundary.

From (1.6) and (1.7), we obtain the inequalities

$$\operatorname{Re}[\rho w_{ii}(x, \nu) + p_{ii}(x)] \leqslant 0 \qquad (2 \leqslant i \leqslant 5), \tag{1.15}$$

which hold for all $x \ge 1$ and $\nu \in B_{1,\alpha}$.

Let j $(2 \le j \le 5)$ be fixed and let $v_{ij}(x, \nu)$ $(2 \le i \le 5)$ be the solution of the Cauchy problem

$$v_{ij} = \delta_{ij} \quad (2 \leqslant i \leqslant 5) \quad \text{at } x = 1$$

for the system (1.14). On the basis of (1.15) we conclude that the functions $v_{ij}(x, \nu)$ $(2 \le i \le 5)$ are uniformly bounded for $x \in (1, \infty)$, $\nu \in B_{1,\alpha}$, $|\rho\nu| \ge N(\alpha)$. The solution v_{ij} $(2 \le i \le 5)$ of system (1.14) corresponds to a solution u_{ij} $(1 \le i \le 5)$ of system (1.5) of the form

$$u_{ij}(x, \nu) = v_{ij}(x, \nu) - u_{i1}(x, \nu) \int_x^\infty u_{11}^{-1}(t, \nu)\, t^{-3/2} \sum_{k=2}^5 q_{1k}(t, \nu)\, v_{kj}(t, \nu)\, dt,$$
$$u_{1j}(x, \nu) = -u_{11}(x, \nu) \times \int_x^\infty u_{11}^{-1}(t, \nu)\, t^{-3/2} \sum_{k=2}^5 q_{1k}(t, \nu)\, v_{kj}(t, \nu)\, dt \quad (2 \leqslant i \leqslant 5). \tag{1.16}$$

Using (1.13) and the boundedness of $v_{ij}(x, \nu)$ for $x \in (1, \infty)$, $\nu \in B_{1,\alpha}$, $|\rho\nu| \ge N(\alpha)$ we obtain from (1.16)

$$u_{ij}(1, \nu, \rho) = \delta_{ij} + O(|\rho\nu|^{-2}),$$
$$u_{1j}(1, \nu, \rho) = O(|\rho\nu|^{-1}) \quad (2 \leqslant i, j \leqslant 5). \tag{1.17}$$

for all $\nu \in B_{1,\alpha}$, $|\rho\nu| \geq N(\alpha)$ for $|\rho\nu| \to \infty$.

In view of (1.9), (1.17) we can represent $S(\nu, \rho)$ in the form

$$S(\nu, \rho) = \rho u_{11}(\nu) + O(1) \tag{1.18}$$

for all $\nu \in B_{1,\alpha}$, $|\rho\nu| \geq N(\alpha)$ for $|\rho\nu| \to \infty$. Note that for all $\nu \in B_{1,\alpha}$ $(\alpha > 0)$ there holds the inequality $|\nu^{-1}u_{11}(\nu)| \geq k(\alpha) > 0$. Taking this into account, we conclude from (1.18) that for any arbitrary $\alpha > 0$ there exists a $\rho_1(\alpha)$ such that for $\rho \geq \rho_1(\alpha)$ there are no eigenvalues in the region $B_{1,\alpha}$.

A second corollary of (1.18) is the following assertion: The eigenvalues $\lambda(\xi_{nk}, s)$ of the problem (0.2)–(0.4) with $\operatorname{Re}\lambda \geq 0$ satisfy the inequality

$$|\lambda| \leqslant c_0 \max(\xi_{nk}, s^{-1}) \tag{1.19}$$

where c_0 is independent of ξ_{nk}, s.

It can be shown that all eigenvalues $\lambda(\xi_{nk}, s)$ satisfy inequality (1.19). To prove this it is sufficient to consider the case $\operatorname{Re}\nu < 0$. The eigenvalues $\nu(\operatorname{Re}\nu < 0)$ are determined from the collinearity condition of the vectors $\mathbf{u}_0(\nu)$ and $\mathbf{u}_1(1, \nu)$, where $\mathbf{u}_1(1, \nu)$ is the unique solution of equation (1.5) which satisfies condition (0.4). The asymptotic representation of this solution for $x \in (1, \infty)$, $|\nu| \geq \nu_0 > 1$, $|\rho\nu| \to \infty$ is of the form (1.16). Comparing (1.9) with (1.16), we find that for large values of $|\rho\nu|$ and $\operatorname{Re}\nu \leq 0$, $|\nu| \geq \nu_0$, the vectors $\mathbf{u}_0(\nu)$ and $\mathbf{u}_1(1, \nu)$ cannot be collinear. This ends the proof of Assertion 1.

§2. Let us compare the results of the mathematical analysis of the stability of a detonation with the experimental facts from the theory of spin detonation. It should be remembered that such a comparison must be carried out very carefully, since the mathematical problem considered here is linear, whereas the phenomenon of spin detonation is essentially nonlinear.

From numerous observations of spin detonation (for example, [3]) it is known that the "number of heads" of the spin increases with increasing tube radius r_0 or initial pressure of the mixture p_0, with all other parameters fixed. These facts follow from a basic empirical law in the phenomenon of spin detonation: The dimension of a "cell" (the characteristic transverse dimension of the structure of the detonation wavefront) is of the order of magnitude of the width of the chemical reaction zone.

In the model considered, this law should be compared with the following fact: For large values of the dimensionless activation energy a (which are characteristic for the majority of detonating gas mixtures), a steady plane detonation wave is unstable with respect to perturbations whose characteristic transverse dimension is of the order of the width of the reaction zone, i.e., $d\xi_{nk}/r_0 \sim 1$.

With increasing r_0 or p_0, the left-hand side of this relation decreases. In order that this relation remain valid (i.e., in order that the condition for instability be satisfied), one must increase the value ξ_{nk}, or, with fixed k, increase the value of n, which is analogous to the number of spin heads.

It should be noted that the author does not have a rigorous proof of the assertion that the basic solution, which represents a steady plane detonation wave, is unstable only with respect to perturbations for which $d\xi_{nk}/r_0 \sim 1$. In order to provide a full answer to this problem, one would have to establish the stability of the basic solution for the case $d\xi_{nk}r_0^{-1} \to 0$. For this, in turn, it would be necessary (but, in general, not sufficient), to prove the stability of the basic solution with respect to one-dimensional perturbations ($\xi_{nk} = 0$). The latter problem can apparently be solved only numerically.

As regards the quantitative agreement between the results of the linear theory and the experimental data, it is not very likely that such agreement can be found in the large. One may expect, however, that the value $|\operatorname{Im}\lambda_n|$ will be close to the value of the dimensionless frequency of n-head spin ω_n. Indeed, calculations show [1] that $|\operatorname{Im}\lambda_n| \sim \sim \xi_{n1}$; on the other hand it is known [3] that the relation $\omega_n = \xi_{n1}$ holds to a high degree of accuracy.

In conclusion, let us touch upon the problem of the possibility of stable propagation of a steady detonation wave.

The author has carried out a numerical computation on an electronic computer in order to find the eigenvalues of the problem (0.2)–(0.4) in the region $\operatorname{Re}\lambda \geq 0$ in the case $a = 0$. The computations, carried out for $\gamma = 1.2$, $c = 0.7925$, $a = 0$, $m = 1$ in a wide range of $\xi_{n1}\delta$

$$0.35 \leqslant \xi_{n1}\delta \leqslant 4.2,$$

showed that there are no eigenvalues λ_n in the region $|\operatorname{Im}\lambda_n| \leq 3.5\xi_{n1}$; $\operatorname{Re}\lambda_n \leq 1.4\xi_{n1}$. (Here the zone width d was defined as the distance over which $\beta = e^{-1}$.) In the region indicated, the function $|F(\lambda)|$ increases with increasing $\operatorname{Im}\lambda$ and increases rapidly with increasing $\operatorname{Re}\lambda$. It is reasonable to assume that even outside the region indicated there will be no eigenvalues λ_n with $\operatorname{Re}\lambda_n \geq 0$.

To complement the numerical analysis of the problem (0.2)–(0.4) in the case $a = 0$, it can be proved that for $a = 0$, $\xi_{nk} > 0$ ($\xi_{nk} = 0$) there are no eigenvalues with $\operatorname{Im}\lambda = 0$, $\operatorname{Re}\lambda \geq 0$ ($\operatorname{Re}\lambda > 0$).

On the basis of the above, one may, with due caution, speak about the stability of the basic solution for $a = 0$. From the stability for $a = 0$ there follows the stability of the basic solution for sufficiently small a. Apparently, the critical value a_*, which determines the limit of stability, is lower than the practical values of a for known detonating gaseous mixtures. In any case, it would be interesting to determine the numerical value of the critical a_*. The solution of this problem would involve, however, very extensive computation.

A second potentially possible stable detonation is detonation in tubes of small radius.

We have proved the stability for the linear approximation of the basic solution in the case $d\xi_{nk}r_0^{-1} \to \infty$ for $c < 1/2$. For $c \geq 1/2$ this result is not quite rigorous. We have no reason to assume that the value $c = 1/2$ is physically exceptional. One may expect that the basic solution will be stable for $d\xi_{nk}r_0^{-1} \to \infty$ also for the case $c \geq 1/2$. Hence, in view of the stability with respect to one-dimensional perturbations ($\xi_{nk} = 0$), it follows that in tubes of sufficiently small radius a steady plane detonation wave is stable according to the linear approximation.

It would be very interesting to establish this fact experimentally. The difficulties which could arise in connection with such an experiment are of two kinds. First, detonation is impossible in tubes with a radius below some critical value. The value of this critical radius is determined by factors which have not been taken into account in our model. Second, stability in the linear approximation does not necessarily lead to stability with respect to finite perturbations.

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27 January 1964 Novosibirsk